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SEMICONTINUOUS REAL NUMBERS IN A TOPOS

Jack Z. REICHMAN
Department of Mathematics, University of Dayton, Dayton, Ohio 45469, USA

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Introduction

It is well known that classically equivalent constructions of the real numbers may yield distinct real number objects (e.g. Dedekind reals, Cauchy reals) when carried out in topoi other than Sets. We introduce a type of real number, which we call a semicontinuous real number, whose construction was suggested by Lawvere [8, 9]. Such a real number is defined by a single closed cut in the rationals. In $\mathrm{Sh}(X)$, for example, this construction yields the sheaf of upper (respectively, lower) semicontinuous functions.

Work of Hofmann [3], Hofmann and Keimel [4], and others indicates that the sheaf of upper semicontinuous functions is the natural recipient of the norm for variable C^*-algebras, variable Banach spaces, and variable metric spaces (also, see Example 1.4 below). In this paper, we study semicontinuous real numbers through examples and by examining the general properties of such objects in an elementary topos; we show that the object of nonnegative upper semicontinuous real numbers has precisely the properties needed by a norm recipient. After defining semicontinuous real numbers and looking at some examples, we show that the semicontinuous reals are an internally complete poset. The existence of an internal 'associated sheaf' functor is demonstrated (Theorem 2.12) and is used in discussing the relationships between the various real number objects. It is also used to define algebraic structures on semicontinuous reals. We conclude with remarks concerning the closed category structure of the semicontinuous reals.

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1. Definition and examples

Let S be an elementary topos with a natural numbers object N. Let Q denote the

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rationals and let Ω denote the subobject classifier in **S**. The power objects Ω^X will be denoted by PX.

Definition. Let $\mu \in PQ$ and let p, q denote generalized elements of Q. μ is an *upper semicontinuous real number* in **S** if μ satisfies the following condition:

$$\forall p \forall q \ ((p<q \Rightarrow q \in \mu) \Leftrightarrow (p \in \mu)). \qquad \text{(usc)}$$

We denote the subobject of upper semicontinuous reals by $i : \mathbb{R}_U \rightarrowtail PQ$. One can similarly define $\mathbb{R}_l$, the object of lower semicontinuous reals.

An equivalent condition defining upper semicontinuous reals (which is used in the following examples) is

$$\forall p \forall q \ \left(\mu(p) = \bigwedge_{q>p} \mu(q)\right). \qquad (1.1)$$

The symbol $\wedge$ used in (1.1) refers to the internal left-adjoint of the $\uparrow$seg map of an internal poset [13].

Example 1.2. If $\mathbf{S} = \text{Sets}$, a closed upper cut of the set of rational numbers is an extended real number ($\pm\infty$ correspond to the instances in which the cut is empty or all of Q).

Example 1.3. Let X be a topological space, U an open subset of X, and let $\bar{\mathbb{R}}$ denote the extended real numbers $[-\infty, \infty]$ with its usual topology. A function $f : U \to \bar{\mathbb{R}}$ is upper semicontinuous iff for each $t \in \bar{\mathbb{R}}$, $\{x \mid f(x) < t\}$ is an open subset of U.

In $\text{Sh}(X)$, let μ be an element of PQ defined over U. We define a function $\tilde{\mu} : U \to \bar{\mathbb{R}}$ as follows, using the identification of Q with locally constant rational valued functions: for each $x \in U$,

$$\tilde{\mu}(x) = \inf\{q(x) : x \in \mu(q) \text{ and } q \in Q(U)\}.$$

Let $t \in \bar{\mathbb{R}}$, and let $x_0 \in \{x : \tilde{\mu}(x) < t\}$. Then, $\tilde{\mu}(x_0) < t$, so by the definition of $\tilde{\mu}$ there must be a rational q_0 such that $q_0(x_0) < t$, $q_0 \in Q(U)$, and $x_0 \in \mu(q_0)$. Let $V = \{x \mid q_0(x) < t\} \cap U \cap \mu(q_0)$. V is an open neighborhood of x_0 and $V \subseteq \{x \mid \tilde{\mu}(x) < t\}$, hence $\tilde{\mu}$ is an upper semicontinuous function on U.

Conversely, given any function $f : U \to \bar{\mathbb{R}}$, define $\tilde{f} \in PQ$ as follows: if V is an open subset of U and $q \in Q(V)$,

$$\tilde{f}_V(q) = \text{the interior of } \{x \in V : f(x) \leq q\}.$$

One can then verify that $\tilde{f}$ satisfies (1.1) and is thus an upper semicontinuous real.

When these operations are restricted to upper semicontinuous functions and upper semicontinuous reals, we obtain a natural isomorphism between $\mathbb{R}_U$ and the sheaf of $\bar{\mathbb{R}}$-valued upper semicontinuous functions.

For a construction which uses inhabited open cuts, see Mulvey [11] or Johnstone [5].

Example 1.4. This example, which is based on the preceding one, illustrates the natural manner in which upper semicontinuous reals arise as norms in a topos. Classically, the norm of a linear map $F: Y \to Z$ between normed linear spaces is defined by a closed upper cut of real numbers (which could just as well be taken as a closed upper cut of rationals):

$$\|F\| = \inf\{M \geq 0: \|F(y)\| \leq M\|y\| \text{ for all } y \in Y\}.$$

Now, if Y and Z are normed linear spaces over X (see [1]), the natural extension of the definition of the norm of a linear map $F: Y \to Z$ gives a continuously variable closed upper cut of Q for each $x \in X$ as the norm of F, i.e. $\|F\|$ is an upper semicontinuous function on X.

Example 1.5. If M is a monoid and $m, n \in M$, we say that $m \leq n$ if there exists an $x \in M$ such that $xm = n$. A function $f: M \to \bar{\mathbb{R}}$ is said to be order-reversing if $m \leq n$ implies $f(m) \geq f(n)$. In the topos of M-Sets (sets with a left M-action), $\mathbb{R}_U$ can be characterized as the set of all order-reversing functions $M \to \bar{\mathbb{R}}$, with M-action defined by $(mf)(n) = f(nm)$. The details are analogous to Example 1.3.

Example 1.6. If P is a poset, let $\tilde{P}$ denote the topological space whose elements are those of P and in which the open sets are the right order-ideals of P. Using the equivalence between $\mathrm{Sh}(\tilde{P})$ and Sets^P, we can describe $\mathbb{R}_U$ in Sets^P as the functor whose value at $p \in P$ is given by $(\mathbb{R}_U)_p = \{f: (p, \to) \to \bar{\mathbb{R}} \mid f \text{ is order-reversing}\}$, where $q \in (p, \to)$ iff $q > p$; the restriction maps are obvious.

In general, the semicontinuous reals will be more closely linked with the topos structure than are the Dedekind reals. For example, in M-Sets, the Dedekind reals are just the constant reals. As another example, consider the fact that while any realcompact space can be recovered from its ring of continuous functions, the open sets themselves are not in general representable by continuous functions. On the other hand, each open set is associated (through its characteristic function) with a multiplicatively idempotent semicontinuous function.

Among the properties of the set of $\bar{\mathbb{R}}$-valued (nonnegative) upper semicontinuous functions on a space X is that it is closed under the pointwise operations on functions of addition, multiplication, finite suprema, and arbitrary infima [2]. These are desirable properties for a norm-recipient. The main body of this paper will be concerned with determining whether (and in which way) these properties extend to the object of upper semicontinuous reals in a topos.

2. Order structure

For each internal poset A in the topos S, we may define the order-reversing morphism $\uparrow\!\mathrm{seg}: A \to PA$; $b \in \uparrow\!\mathrm{seg}\, a$ iff $a \leq b$. An internal poset A is internally complete if $\uparrow$seg has an internal left-adjoint $\inf : PA \to A$. Equivalently, A is internally complete if $\downarrow\text{seg} : A \to PA$ has an internal left-adjoint $\sup : PA \to A$ (Mikkelsen [10]). If A is internally complete, then so is A^X for any object X of **S**. In particular, Ω- and hence PX- is an internally complete poset in any topos. The internal left-adjoint to $\uparrow\text{seg} : PX \to PPX$ shall be designated by $\bigcap$. Note that for any $q \in Q$, $\uparrow\text{seg}\, q \in \mathbb{R}_U$.

We give $\mathbb{R}_U$ the order which it inherits as a subobject of PQ, i.e., if $\phi, \psi \in \mathbb{R}_U$ then $\phi \le \psi$ iff $i\phi \le i\psi$. If $q \in Q$, let $\alpha q \in PQ$ be defined by $p \in \alpha q$ iff $p > q$. Restated, the condition for $\mu \in PQ$ to be upper semicontinuous is

$$q \in \mu \quad \text{iff} \quad \alpha q \le \mu. \tag{2.1}$$

In the ensuing proofs, we shall use tools such as the 'existence principle' to interpret various statements; full details may be found in [5], [6], and [10].

Lemma 2.2. *Let $F : X \to P\mathbb{R}_U$ be a morphism of* **S**. *Let F^* denote the composite*

$$X \xrightarrow{F} P\mathbb{R}_U \xrightarrow{\exists i} PPQ \xrightarrow{\bigcap} PQ.$$

Then, F^* is an upper semicontinuous real in **S**.

Proof. Using (2.1), F^* is upper semicontinuous iff for every $x : Y \to X$ and $q : Y \to Q$, $q \in F^*x$ iff $\alpha q \le F^*x$. If x, q are as above, then for every $y : W \to Y$ and $A : W \to PQ$, we have

$$q \in F^*x \quad \text{iff} \quad A \in \exists i(Fxy) \quad \text{implies} \quad qy \in A, \tag{2.3}$$

and

$$\alpha q \le F^*x \quad \text{iff} \quad A \in \exists i(Fxy) \quad \text{implies} \quad \alpha qy \le A. \tag{2.4}$$

We first show that given $\langle x, q\rangle : Y \to X \times Q$, $q \in F^*x$ implies $\alpha q \le F^*x$. Let $\langle y, A\rangle : W \to Y \times PQ$ be given, and assume that $A \in \exists i(Fxy)$. From the existence principle, there is an epic $\beta : V \twoheadrightarrow Q$ and a $\phi : V \to \mathbb{R}_U$ such that $\phi \in Fxy\beta$ and $i\phi = A\beta$. From (2.3) above, $A \in \exists i(Fxy)$ gives $qy \in A$ and hence $qy\beta \in A\beta$. Since $A\beta = i\phi$, $qy\beta \in i\phi$. But $i\phi$ is upper semicontinuous, so $qy\beta \in i\phi$ implies $\alpha qy\beta \le i\phi = A\beta$. Since β is epic, $\alpha qy \le A$. Using (2.4), this shows that $\alpha q \le F^*x$.

Conversely, assume that $\alpha q \le F^*x$, and that A, y are given as above, with $A \in \exists i(Fxy)$. As before, we obtain an epic β and an upper semicontinuous ϕ such that $\phi \in Fxy\beta$ and $i\phi = A\beta$. Since $\alpha q \le F^*x$ and $A \in \exists i(Fxy)$, we obtain (by (2.4)) $\alpha qy \le A$, and hence $\alpha qy\beta \le A\beta = i\phi$. The upper semicontinuity of $i\phi$ gives $qy\beta \in A\beta$, hence $qy \in A$, hence $q \in F^*x$ by (2.3). $\square$

The preceding lemma shows that $\bigcap \circ \exists i : P\mathbb{R}_U \to PQ$ is itself an upper semicontinuous real, hence it factors through $i : \mathbb{R}_U \to PQ$. Let $\text{Inf} : P\mathbb{R}_U \to \mathbb{R}_U$ denote the morphism such that $i \circ \text{Inf} = \bigcap \circ \exists i$. Since i and $\exists i$ are convariant internal functors, and since $\bigcap$ is contravariant, Inf is contravariant.

Theorem 2.5. *$\mathbb{R}_U$ is an internally complete poset.*

Proof. We will demonstrate that Inf is internally left-adjoint to $\uparrow\mathrm{seg} : \mathbb{R}_U \to P(\mathbb{R}_U)$, i.e. if $F : X \to P(\mathbb{R}_U)$ and $\phi : X \to \mathbb{R}_U$, then

$$\phi \leq \mathrm{Inf}(F) \quad \text{iff} \quad F \leq \uparrow\mathrm{seg}\,\phi. \tag{2.6}$$

Before proceeding, note that for every $x : Y \to X$ and $\mu : Y \to PQ$,

$$\phi \leq \mathrm{Inf}(F) \quad \text{iff} \quad \mu \in \exists i(Fx) \quad \text{implies} \quad i\phi x \leq \mu, \tag{2.7}$$

and

$$F \leq \uparrow\mathrm{seg}\,\phi \quad \text{iff} \quad \mu \in Fx \quad \text{implies} \quad \mu \in \uparrow\mathrm{seg}\,\phi x. \tag{2.8}$$

First suppose $\phi \leq \mathrm{Inf}(F)$. A trivial application of the existence principle shows that $\mu \in Fx$ implies $i\mu \in \exists i(Fx)$. By (2.7), $i\mu \in \exists i(Fx)$ implies $i\phi x \leq i\mu$. But $i\phi x \leq i\mu$ iff $\phi x \leq \mu$, and $\phi x \leq \mu$ iff $\mu \in \uparrow\mathrm{seg}\,\phi x$.

Conversely, assume $F \leq \uparrow\mathrm{seg}\,\phi$ and that $\mu \in PQ$ is given, with $\mu \in \exists i(Fx)$. Since $\mu \in \exists i(Fx)$, there is an epic $\beta : W \twoheadrightarrow Y$ and a $\psi : W \to \mathbb{R}_U$ such that $\psi \in Fx\beta$ and $i\psi = \mu\beta$. By (2.8), $\psi \in \uparrow\mathrm{seg}\,\phi x\beta$, hence $\phi x\beta \leq \psi$; hence $i\phi x\beta \leq i\psi = \mu\beta$, so $i\phi x \leq \mu$. Thus $\phi \leq \mathrm{Inf}(F)$ by (2.7), which shows that Inf and $\uparrow$seg are adjoint. □

Corollary 2.9. *$\downarrow\mathrm{seg} : \mathbb{R}_U \to P(\mathbb{R}_U)$ has an internal left-adjoint, denoted* Sup : $P(\mathbb{R}_U) \to \mathbb{R}_U$.

The internal completeness of $\mathbb{R}_U$ leads to an internal associated sheaf functor $L : PQ \to \mathbb{R}_U$. A morphism $f : A \to B$ between internally complete posets is said to be inf-preserving if the following diagram commutes:

$$\begin{array}{ccc} PA & \xrightarrow{\exists f} & PB \\ {\scriptstyle \mathrm{inf}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{inf}_B} \\ A & \xrightarrow{f} & B \end{array} \tag{2.10}$$

Lemma 2.11. *Let A and B be internally complete posets and let $f : A \to B$. Then, f is inf-preserving iff f is an internal functor and has a left-adjoint g which is sup-preserving.*

Proof. See Mikkelsen [10].

Theorem 2.12. *$i : \mathbb{R}_U \rightarrowtail PQ$ has an internal left-adjoint $L : PQ \to \mathbb{R}_U$. Moreover, L is epic and is sup-preserving.*

Proof. Inf was defined so that diagram (2.10) commutes, hence i is inf-preserving, hence the desired L exists by the preceding lemma. L is epic because i is monic and L is internally left-adjoint to i. □

The next proposition summarizes some easy consequences of the internal adjointness of L and i.

Proposition 2.13. *If $\mu \in PQ$ and $\phi \in \mathbb{R}_U$, then*
(i) *$iL\mu$ is the least upper semicontinuous real which dominates μ,*
(ii) *$Li\phi = \phi$, and*
(iii) *$iL\mu = \mu$ iff μ is upper semicontinuous.*

Proof. (i) From adjointness, $1_{PQ} \leq iL$, so $\mu \leq iL\mu$. For the same reason, $\mu \leq i\phi$ iff $L\mu \leq \phi$.

(ii) Since $i\phi \leq i\phi$, adjointness gives $Li\phi \leq \phi$ and $i\phi \leq iL(i\phi)$. But $i\phi \leq iL(i\phi)$ iff $\phi \leq Li\phi$.

(iii) If μ is upper semicontinuous, we can write $\mu = i\psi$ for some $\psi : X \to \mathbb{R}_U$. Then, $iL\mu = i(Li\psi) = i\psi$ by (ii), hence $iL\mu = \mu$. Conversely, $iL\mu = \mu$ trivially implies $\mu = i(L\mu)$, and so μ factors through $\mathbb{R}_U$. □

The above results generalize the fact that if $\{f_i : X \to \bar{\mathbb{R}}\}$ is a family of upper semicontinuous functions on a space X, then $f = \inf_i f_i$ (pointwise infimum) is an upper semicontinuous function. $L : PQ \to \mathbb{R}_U$ is related to the *upper semicontinuous regularization* $\bar{f}$ of an arbitrary function $f : X \to \bar{\mathbb{R}}$; the regularization $\bar{f}$ is defined for each $x \in X$ by

$$\bar{f}(x) = \limsup_{y \to x} f(y).$$

$\bar{f}$ is upper semicontinuous and is the least upper semicontinuous function g such that $f \leq g$, i.e. upper semicontinuous regularization is left-adjoint to the inclusion of usc$(X, \bar{\mathbb{R}})$ in the set of all functions $X \to \bar{\mathbb{R}}$ (where both are categories with $f \to g$ iff $f \leq g$).

Arbitrary suprema of upper semicontinuous functions can be calculated using the regularization: if $f = \sup_i f_i$ is the pointwise supremum, then $\bar{f}$ is the least upper bound (among the upper semicontinuous functions). In general, the internal completeness of $\mathbb{R}_U$ makes it a more useful real number object (as a norm-recipient) than the Dedekind reals in that the greatest lower bound (and l.u.b.) of families of 'real numbers' always exists.

The next result points out one use of the internal regularization functor L.

Proposition 2.14. *If $\phi, \psi : X \to PQ$ are upper semicontinuous, then the upper semicontinuous binary meet and join of ϕ and ψ are $\phi \wedge \psi$ and $iL(\phi \vee \psi)$, respectively.*

Proof. First, let $x : Y \to X$ and $q : Y \to Q$ be given. Then, $q \in \phi x \wedge \psi x$ iff $q \in \phi x$ and $q \in \psi x$ iff $\alpha q \leq \phi x$ and $\alpha q \leq \psi x$ iff $\alpha q \leq \phi x \wedge \psi x$. Hence $\phi \wedge \psi$ is upper semicontinuous; it is clearly the greatest element of $\mathbb{R}_U$ which is dominated by both ϕ and ψ because $\mathbb{R}_U$ is a subobject of PQ.

Second, $iL(\phi\vee\psi)$ is the least upper semicontinuous real which dominates $\phi\vee\psi$ (by 2.13(i)), $\phi\vee\psi$ is the least element of PQ which dominates both ϕ and ψ, and hence $iL(\phi\vee\psi)$ is the upper semicontinuous binary join of ϕ and ψ. □

In many instances, we will be considering a $\mu\in PQ$ which is known to be an upper cut, i.e. μ satisfies the following condition:

$$\forall p \forall q \ ((p<q \wedge p\in\mu) \ \Rightarrow \ (q\in\mu)), \tag{2.15}$$

In other words, $p\in\mu \ \Rightarrow \ \alpha p\leq\mu$. The next result provides a useful description of $L\mu$ for those $\mu\in PQ$ which satisfy (2.15).

Lemma 2.16. *If $\mu : X\to PQ$, then the following are equivalent:*
(i) *μ satisfies* (2.15).
(ii) *for every $\langle x,q\rangle : Y\to X\times Q$, $q\in iL\mu$ iff $\alpha q\leq\mu x$.*

Proof. (ii)⇒(i) follows directly from the fact that $\mu\leq iL\mu$. Conversely, assume (i) is satisfied. Define $J\mu : X\to P$ such that for every $\langle x,q\rangle : Y\to X\times Q$, $q\in J\mu x$ iff $\alpha q\leq\mu x$. Now, $\mu\leq J\mu$. Moreover, $J\mu$ is upper semicontinuous. Since $\mu\leq J\mu$ and $J\mu$ is upper semicontinuous, we obtain $iL\mu\leq J\mu$ (by 2.13(i)). On the other hand, if $q\in J\mu x$, then $\alpha q\leq\mu x\leq iL\mu x$. Since $iL\mu$ is upper semicontinuous, $\alpha q\leq iL\mu x$ implies $q\in iL\mu$; hence $J\mu\leq iL\mu$. Thus, $J\mu = iL\mu$. □

3. Other real number objects and the semicontinuous reals

The roster of real number objects in a topos includes the Cauchy reals, the Dedekind reals, and the Dedekind–MacNeille reals, in addition to the upper (and lower) semicontinuous reals. What relationships, if any, exist among these real number objects? For example, it is well known that in Sh(X), the Dedekind reals can be characterized as the sheaf of continuous real-valued functions [11]. Since every continuous function is upper semicontinuous, is it generally true that the Dedekind reals will be a subobject of $\mathbb{R}_U$? The answer is yes, which we now proceed to show.

Informally, a Dedekind real number is a pair $\langle\lambda,\mu\rangle$ of elements of Ω^Q such that λ (respectively, μ) is an inhabited, open lower (upper) cut, and such that λ and μ are adjacent and disjoint. The conditions that $\langle\lambda,\mu\rangle$ must satisfy are usually formulated in the internal language of the topos. These conditions are:

(DR 1) $q\in\lambda \ \Leftrightarrow \ \exists p(p>q\wedge p\in\lambda)$
(DR 2) $q\in\mu \ \Leftrightarrow \ \exists p(p<q\wedge p\in\mu)$
(DR 3) $(q\in\lambda\wedge p\in\mu) \ \Rightarrow \ q<p$
(DR 4) $q<p \ \Rightarrow \ (q\in\lambda\vee p\in\mu)$
(DR 5) $\exists q(q\in\lambda)\wedge\exists p(p\in\mu)$.

Let $\mathbb{R}_D \rightarrowtail \Omega^Q\times\Omega^Q$ denote the object of Dedekind reals. The properties of $\mathbb{R}_D$ are examined in [5] and in [12] (where they are called the continuous reals).

Lemma 3.1. *Let $r_1=(l_1,u_1)$, $r_2=(l_2,u_2)$ be Dedekind reals.*
(i) $l_1 \le l_2$ *iff* $u_2 \le u_1$.
(ii) $r_1=r_2$ *iff* $\mu_1=\mu_2$.

Proof. The proof of (i) may be found in Johnstone [5], lemma 6.63(ii). The second part of our lemma follows directly from (i). □

Proposition 3.2. *$\mathbb{R}_D$ is a subobject of $\mathbb{R}_U$.*

Proof. We define $f:\mathbb{R}_D \to \mathbb{R}_U$ to be the composite

$$\mathbb{R}_D \rightarrowtail \Omega^Q \times \Omega^Q \xrightarrow{\pi_2} \Omega^Q \xrightarrow{L} \mathbb{R}_U.$$

We claim that f is monic, i.e. if $r_1,r_2 : X \to \mathbb{R}_D$ and $fr_1=fr_2$, then $r_1=r_2$. Let $r_i=(l_i,u_i)$; the assumption $fr_1=fr_2$ means $Lu_1=Lu_2$. If we demonstrate that $u_1=u_2$, then we will have $r_1=r_2$ by the preceding lemma. Note that u_i satisfies (2.15) because of condition (DR 2) for Dedekind reals. Hence, using the characterization in Lemma 2.16 of $L\mu$ for those μ which satisfy (2.15), we have for any $\langle x,q\rangle : Y \to X\times Q$,

$$\begin{aligned} \alpha q \le u_1 x \quad &\text{iff} \quad q \in Lu_1 x \\ &\text{iff} \quad q\in Lu_2 x \\ &\text{iff} \quad \alpha q \le u_2 x. \end{aligned} \qquad (*)$$

We use the above to show that $u_1 \le u_2$. Suppose $\langle x,q\rangle : Y \to X\times Q$ and that $q\in u_1 x$. By (DR 2), there is an epic $\beta : V \twoheadrightarrow Y$ and a $p: V\to Q$ such that $p<q\beta$ and $p\in u_1 x\beta$. Since $u_1 x\beta$ satisfies (2.15), $p\in u_1 x\beta$ implies $\alpha p \le u_1 x\beta$. By (*), we then have $\alpha p\le u_2 x\beta$. Since $p<q\beta$, we thus have $q\beta \in u_2 x\beta$ but β is epic, so $q\in u_2 x$. Applying the extensionality principle, this shows $u_1\le u_2$. Similarly $u_1 \le u_2$, hence $u_1=u_2$, $r_1=r_2$, and f is monic. □

Corollary 3.3. *$\mathbb{R}_C$, the object of 'Cauchy reals' (as defined in 6.67 of [5]) is a subobject of $\mathbb{R}_U$.*

Proof. It is shown in [5] and [12] that $\mathbb{R}_C$ is a subobject of $\mathbb{R}_D$. Applying Proposition 3.2, we are done. □

We note that $\mathbb{R}_l$ is isomorphic to $\mathbb{R}_U$: if $\mu\in\mathbb{R}_U$, define $-\mu$ by $q\in -\mu$ iff $-q\in\mu$; and similarly for $\lambda \in \mathbb{R}_l$. It is clear that $-\mu\in\mathbb{R}_l$ iff $\mu\in\mathbb{R}_U$ and that $-\mu_1=-\mu_2$ iff $\mu_1=\mu_2$; hence the desired isomorphism.

4. Algebraic structures

We say $\mu \in PQ$ is *nonnegative* if $\mu \le \uparrow\text{seg}\,0$ ($0=$ additive identity of Q). $\mathbb{R}_U^+$ will denote the object of nonnegative upper semicontinuous reals.

Definition 4.1. Let $\mu_1, \mu_2 \in PQ$ (with a common domain). Define $\mu_1 + \mu_2$ as follows: if $p \in Q$, $p \in \mu_1 + \mu_2$ iff $\exists p_1 \exists p_2\, (p_1 \in \mu_1 \wedge p_2 \in \mu_2 \wedge p = p_1 + p_2)$. We denote this binary operation on PQ by $+ : PQ \times PQ \to PQ$.

Some properties of + are summarized in the next lemma. The proof is straightforward and is left to the reader.

Lemma 4.2. (i) $+ : PQ \times PQ \to PQ$ *is commutative.*

(ii) *If* $\mu, \phi \in PQ$, μ *satisfies* (2.15), *and* ϕ *is nonnegative, then* $\mu + \phi \leq \mu$.

(iii) *If* $\mu \in PQ$ *satisfies* (2.15), *then* $\mu + \uparrow\text{seg}\, 0 = \mu$.

(iv) *If* $\mu, \phi \in PQ$ *and* ϕ *satisfies* (2.15), *then* $\mu + \phi$ *satisfies* (2.15).

Definition. Addition for upper semicontinuous reals, $\oplus$, is the composite

$$\mathbb{R}_U \times \mathbb{R}_U \xrightarrow{i \times i} PQ \times PQ \xrightarrow{+} PQ \xrightarrow{L} \mathbb{R}_U.$$

Theorem 4.3. $(\mathbb{R}_U, \oplus)$ *is a commutative monoid in* **S**.

Proof. Let μ, μ_1, μ_2, μ_3 denote elements of $\mathbb{R}_U$. Since $\mu_1 \oplus \mu_2 = L(i\mu_1 + i\mu_2) = L(i\mu_2 + i\mu_1) = \mu_2 \oplus \mu_1$, $\oplus$ is commutative. The identity for $\oplus$ is $\uparrow\text{seg}\, 0 : 1 \to PQ$; since every $\mu \in \mathbb{R}_U$ satisfies (2.15), we see from 4.2(iii) that $\mu + \uparrow\text{seg}\, 0 = \mu$.

Lastly, we must show that addition is associative, i.e. $(\mu_1 \oplus \mu_2) \oplus \mu_3 = \mu_1 \oplus (\mu_2 \oplus \mu_3)$. Observe that by 4.2(iv), $i(\mu_1 \oplus \mu_2) + i\mu_3$ satisfies (2.15). Since $(\mu_1 \oplus \mu_2) \oplus \mu_3 = L[i(\mu_1 \oplus \mu_2) + i\mu_3]$, we apply Lemma (2.16) to obtain

$$q \in i[(\mu_1 \oplus \mu_2) \oplus \mu_3] \quad \text{iff} \quad \alpha q \leq i(\mu_1 \oplus \mu_2) + i\mu_3. \tag{4.4}$$

Similarly,

$$q \in i[\mu_1 \oplus (\mu_2 \oplus \mu_3)] \quad \text{iff} \quad \alpha q \leq i\mu_1 + i(\mu_2 \oplus \mu_3). \tag{4.5}$$

So, assume $\alpha q \leq i(\mu_1 \oplus \mu_2) + i\mu_3$. Assume $p > q$, and let $p' = \frac{1}{2}(p+q)$, so that $p > p' > q$. Since $p' > q$, $p' \in \alpha q$, hence $p' \in i(\mu_1 + \mu_2) \oplus i\mu_3$, and hence $\exists r' \exists p_3$ $(r' \in i(\mu_1 \oplus \mu_2) \wedge p_3' \in i\mu_3 \wedge p' = r' + p_3)$. Applying Lemma 2.16 again, $r' \in i(\mu_1 \oplus \mu_2)$ implies $\alpha r' \leq i\mu_1 + i\mu_2$. Now, let $t = p - p'$; then $t > 0$ and $p = p' + t = r' + p_3 + t = r' + t + p_3$. Since $t > 0$, $r' + t > r'$, hence $r' + t \in i\mu_1 + i\mu_2$. Hence, $\exists p_1 \exists p_2$ $(p_1 \in i\mu_1 \wedge p_2 \in i\mu_2 \wedge r' + t = p_1 + p_2)$. Thus $p = r' + t + p_3 = (p_1 + p_2) + p_3 = p_1 + (p_2 + p_3)$. Now, $p_2 + p_3 \in i\mu_2 + i\mu_3$ and therefore $p_2 + p_3 \in i(\mu_2 \oplus \mu_3)$. It follows that $p_1 + (p_2 + p_3) \in i\mu_1 + i(\mu_2 \oplus \mu_3)$.

Summarizing, we have demonstrated that if $\alpha q \leq i(\mu_1 \oplus \mu_2) + i\mu_3$ and $p > q$, then $p \in i\mu_1 + i(\mu_2 \oplus \mu_3)$. Hence $\alpha q \leq i(\mu_1 \oplus \mu_2) + i\mu_3$ implies $\alpha q \leq i\mu_1 + i(\mu_2 \oplus \mu_3)$. Using (4.4) and (4.5), this shows that $i[(\mu_1 \oplus \mu_2) \oplus \mu_3] \leq i[\mu_1 \oplus (\mu_2 \oplus \mu_3)]$. The other inequality may be derived analogously, hence $\oplus$ is associative. $\square$

If $\mu_1, \mu_2 \in \mathbb{R}_U^+$, then $\uparrow\text{seg}\, 0 \leq i\mu_1$, $i\mu_1 \leq i\mu_1 + i\mu_2$ (by 4.2(ii)), $i\mu_1 + i\mu_2 \leq i(\mu_1 \oplus \mu_2)$ (by

2.13(i)), and hence $\mu_1 \oplus \mu_2 \in \mathbb{R}_U^+$. Thus $\mathbb{R}_U^+$ is also an internal monoid under the inherited operation of $\oplus$. Moreover, $\mathbb{R}_U^+$ is also internally complete; to keep the notation simple, we will denote the inclusion of $\mathbb{R}_U^+$ in PQ by i, and its left adjoint by L.

Besides the addition on $\mathbb{R}_U^+$, we shall define a multiplication $\odot : \mathbb{R}_U^+ \times \mathbb{R}_U^+ \to \mathbb{R}_U^+$. Of necessity, multiplication must be restricted to $\mathbb{R}_U^+$ (e.g. if f is an upper semicontinuous function, $-f$ is lower semicontinuous).

The definition of $\odot$ is analagous to that of $\oplus$. Start by defining, for $\mu_1, \mu_2 \in PQ$, $\mu_1 \cdot \mu_2$ by

$$p \in \mu_1 \cdot \mu_2 \quad \text{iff} \quad \exists p_1 \exists p_2 \; (p_1 \in \mu_1 \wedge p_2 \in \mu_2 \wedge p = p_1 + p_2).$$

Then, if μ_1 and μ_2 are nonnegative and satisfy (2.15), so does $\mu_1 \cdot \mu_2$. For $\mu_1, \mu_2 \in \mathbb{R}_U^+$, define $\mu_1 \odot \mu_2$ by $\mu_1 \odot \mu_2 = L(i\mu_1 \cdot i\mu_2)$. The proof of the next theorem is similar to that of Theorem 4.3 and is left to the reader.

Theorem 4.6. (i) *($\mathbb{R}_U^+$, $\odot$) is an internal commutative monoid with identity* $\uparrow$seg 1.

(ii) (Distributive law.) *If $\mu_1, \mu_2, \mu_3 \in \mathbb{R}_U^+$, then*

$$\mu_1 \odot (\mu_2 \oplus \mu_3) = (\mu_1 \odot \mu_2) \oplus (\mu_1 \odot \mu_3).$$

The internal completeness of $\mathbb{R}_U$ and $\mathbb{R}_U^+$ allows one to define a closed category structure on the semicontinuous reals. For example, let USC(X) denote the set of upper semicontinuous functions (valued in $[0, \infty]$) on a topological space X. As a category, $f \to g$ iff $f \geq g$ (pointwise). If $f, g \in$ USC(X), define $f \otimes g = f + g$ (pointwise sum) and define

$$\hom(f, g) = \inf\{h \in \mathrm{USC}(X) \mid g \leq f + h\} \quad \text{(pointwise inf)}.$$

USC(X) is a closed category with this $\otimes$ and hom.

In general, one can similarly define a closed structure on $\mathbb{R}_U^+$ in any elementary topos $\mathbf{S}$ with a natural number object. Since $\mathbb{R}_U^+ = [0, \infty]$ in $\mathbf{S}$ = Sets, this is a generalization of the closed category described by Lawvere in [7]. In the most general sense then, $\mathbb{R}_U^+$ possesses all of the desirable properties of a metric-recipient. An investigation of internal $\mathbb{R}_U^+$-valued categories and other questions related to the closed structure of $\mathbb{R}_U^+$ will appear separately in a forthcoming paper.

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